SS

NST Maths:

I:

$[\underline{a},\underline{b},\underline{c}] = \underline{a}\cdot(\underline{b}\times\underline{c})$ = volume of parallelepiped invariant under cyclic permutations, else negative.

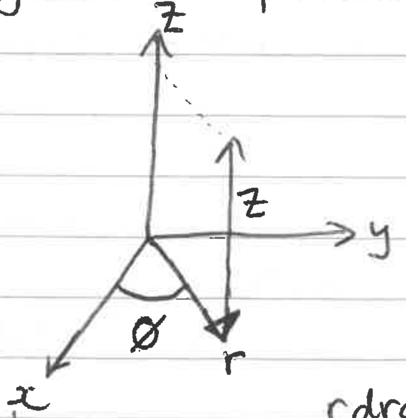
$\underline{a}\times(\underline{b}\times\underline{c}) = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}$

~~Reciprocal bases:~~ $[a,b,c]\neq 0 \Rightarrow$ not coplanar $\Rightarrow$ form a basis. Extract components with reciprocal bases $\lambda = \underline{r}\cdot\underline{A}$ where $\underline{A} = \dfrac{b\times c}{[a,b,c]}$

Basis right handed iff $[a,b,c] > 0$.

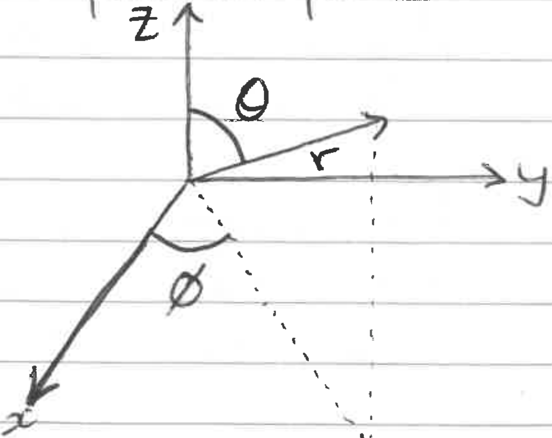
Vector area, $\underline{A} = A\hat{\underline{n}}$ where A is the area and $\hat{\underline{n}}$ is the unit normal. 0 for closed surface. projection on a given plane = normal of plane $\cdot\,\underline{A}$.

Cylindrical polars:



volume element = ~~rdφdzdr~~ $r\,dr\,d\phi\,dz$

Spherical polars



volume element $= r^2\sin\theta\,dr\,d\theta\,d\phi$

Fundamental theory of algebra: an n^{th} degree polynomial has n complex roots exactly.

$$\ln(z) = \ln(re^{i\theta}) = \ln(r) + \ln e^{i\theta} = \ln(r) + i(\theta + 2n\pi)$$

$$z_1^{z_2} \equiv e^{z_2 \ln z_1}$$

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \qquad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

$$\cosh x = \cos(ix) \qquad i\sinh(x) = \sin(ix)$$

To be differentiable a curve must be both continuous and smooth.

Leibnitz's rule: $\frac{d^n (fg)}{dx^n} = \sum_{m=0}^{n} \binom{n}{m} f^{(n-m)} g^{(m)}$

Curvature (d^2y/dx^2) changes sign at inflection points.

$$|z| = \sqrt{zz^*} = \sqrt{a^2+b^2}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\cosh^2 x - \sinh^2 x = 1$$

Algebra of limits is ~~ab~~ as expected for addition and multiplication.

For division if numerator $=0$ and denominator $\neq 0$ then limit is 0, if both are 0 or both are $\pm\infty$ then use l'Hôpital.

$$\lim_{x\to x_0}\left(\frac{f}{g}\right) = \lim_{x\to x_0}\left(\frac{f'}{g'}\right)$$

Function is continuous at x_0 if $f(x_0)$ exists and $\lim_{x\to x_0} f(x) = f(x_0)$

$\sum_{K=0}^{\infty} U_K$ converges to S if $\lim_{n\to\infty}\left(\sum_{K=0}^{n} U_K\right) = S$

Absolutely convergent if $\sum_{K=0}^{n} |U_K|$ converges, otherwise conditionally convergent (or not convergent at all).

Geometric sequence is absolutely convergent.

Checking for convergence: if $\lim_{K\to\infty} U_K \neq 0$ then diverges, else check if we have an oscillating sequence, if so it diverges, else use ratio test, ~~ot~~ if this is inconclusive use comparison test (with $\sum_{n=0}^{\infty} \frac{1}{n^p}$ which converges for only $p > 1$).

Ratio test $\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right|$ if < 1 converges, if > 1 diverges if 1 then inconclusive.

Comparison test: check if $U_K \leq V_K$ or $U_K \geq V_K$ $\forall K >$ some K.

Taylor Series:

$$f(x) = f(a) + (x-a) f'(a) + \dots + (x-a)^K f^{(K)}(a)$$

Taylor's Theorem:

For an n-times differentiable function:

$$f(x) = f(0) + x f^{(1)}(0) + \frac{x^2}{2!} f^{(2)}(0) + \dots + \underbrace{\frac{x^n}{n!} f^{(n)}(S)}_{\equiv R_n}$$

where $0 \leq S \leq x$ and R_n is the remainder.

If $\lim_{n\to\infty} R_n = 0$ the infinite taylor series must converge to $f(x)$.

Newton-Raphson: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $(e_{i+1} \propto e_i^2)$

Integration tricks:

Inspection (duh).

Substitution.

Trig substitutions:

- $\sqrt{a^2 - x^2}$ $\quad x = a\sin\theta$ or $x = a\cos\theta$
- $a^2 + x^2$ $\quad x = a\tan\theta$

Hyporbolic substitutions:

- $\sqrt{a^2+x^2}$ $\quad x = a\sinh y$ ~~or $x = a\cosh y$~~
- $\sqrt{a^2-x^2}$ $\quad x = a\cosh y$
- $a^2 - x^2$ $\quad x = a\tanh y$ if $|x| < a$ else use $x = a\coth y$

Partial fractions.

Complete the square.

Parts.

Trig identities (e.g. $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$)

Complex numbers (for exponentials and trigs).

Symmetry.

~~Partial~~ ~~d~~Derivative of an integral wrt parameter:

let $I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\, dx$

$$\frac{dI}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f(x,\alpha)}{\partial \alpha} dx + \frac{db}{d\alpha} f(b,\alpha) - \frac{da}{d\alpha} f(a,\alpha)$$

Stirling's Approximation: $\ln(n!) \simeq n\ln n - n$.

but $n! \approx n^n e^{-n} \sqrt{2\pi n}$ better approximation for $n!$.

Schwartz's inequality: $\left(\int_a^b f(x) g(x)\, dx\right)^2 \leqslant \int_a^b f^2(x)dx \int_a^b g^2(x) dx$

Area element in plane polars is $r\,dr\,d\phi$

$$\iint_S f(x)g(y)\,dx\,dy = \int_c^d g(y)\,dy \int_a^b f(x)\,dx$$

Gaussian integral: $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

$P(A \cup B) = P(A) + P(B)$ if mutually exclusive.

$P(A \cap B) = P(A)\,P(B)$ if independent.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

counted in both $P(A)$ and $P(B)$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Probability distributions must be normalised, i.e. sum of probabilities is 1.

II:

Ordinary differential equations:

- Separable: separate and integrate.
- Homogeneous: if when x and y are replaced with αx and αy the αs cancel then homogeneous, sub in $y = ux$ when u is a function of x.
- Exact: if LHS is $\frac{d}{dx}$ of something and RHS is func^n of x then just integrate.
- Integrating Factor: $\frac{dy}{dx} + p(x)y = q(x)$, use $e^{\int p(x)dx}$ to make it exact.

- Bernoulli: $\frac{dy}{dx} + p(x)y = y^n q(x)$, sub $z = y^{1-n}$ to make it linear.
- If none of those work a substitution is needed.

Second order differential equations:

general solution = complementary function + particular integral

complementary function: RHS $=0$ solve for λ_1, λ_2:

if distinct real: $Ae^{\lambda_1 x} + Be^{\lambda_2 x}$

if distinct complex $(a \pm ib)$: $e^{ax}(A\cos(bx) + B\sin(bx))$

if equal roots: $(Ax+B)e^{\lambda x}$

If particular integral attempt contains a function that is part of the CF then try multiplying by x or x^2.

Chain rule: if $F = f(x,y)$ then $df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$ is always true. (Can be interpreted as any partial or real derivative)

f_{xx} represents $\frac{\partial^2 f}{\partial x^2}$, note that $f_{xy} = f_{yx}$

Chain rule of partial differentiation:

$$\left(\frac{\partial f}{\partial u}\right)_v = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial u}\right)_v$$

$$\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x = -1 \qquad \left(\frac{\partial y}{\partial x}\right)_z \left(\frac{\partial x}{\partial y}\right)_z = 1$$

$Pdx + Qdy$ is exact if $\left(\frac{\partial P}{\partial y}\right)_x = \left(\frac{\partial Q}{\partial x}\right)_y$

Reciprocity: $\left(\frac{\partial x}{\partial y}\right)_z = 1 \Big/ \left(\frac{\partial y}{\partial x}\right)_z$

Cyclic: $\left(\frac{\partial x}{\partial z}\right)_y = -\left(\frac{\partial y}{\partial z}\right)_x \Big/ \left(\frac{\partial y}{\partial x}\right)_z$

Integrating factors:
- Note that $p(x)\,dx + q(y)\,dy$ is always exact and guess $\mu(x)$ or $\mu(y)$ for integrating factors.
- Otherwise use formulae:

For $\mu(x)$: $\frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

For $\mu(y)$: $\frac{1}{\mu}\frac{d\mu}{dy} = \frac{-1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

μ only exists if RHS is in terms of whichever of x or y μ is assumed to be in terms of.

Maxwell's Relations: $dU = TdS - pdV$ is a statement of the first law of thermodynamics. U is internal energy.

Re U as function of (S,V): $dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV$

$\therefore$ $T = -p$, & now do $\frac{\partial}{\partial V}$ and $\frac{\partial}{\partial S}$ $\frac{\partial^2 U}{\partial S \partial V} = \frac{\partial^2 U}{\partial V \partial S} \Rightarrow \left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V$

Can derive similar relations by inventing new functions, i.e. $F = U - TS$, $H = U + pV$, $G = H - TS$

"By regarding z as a function of x and y" means either / both of:

- interpret differentials as $\partial/\partial x$ and $\partial/\partial y$
- use chain rule: $dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$

To get an expression for $\left(\frac{\partial y}{\partial x}\right)_a - \left(\frac{\partial y}{\partial x}\right)_b$ use

chain rule $dy = \left(\frac{\partial y}{\partial x}\right)_b dx + \left(\frac{\partial y}{\partial b}\right)_x db$

and interpret as $(\partial/\partial x)_a$

Stationary points:

$f_x = f_y = 0 \Rightarrow$ stationary point

if $f_{xx} f_{yy} > f_{xy}^2$ then if $f_{xx} > 0$ its a minimum, if ~~f_{yy}~~ $f_{xx} < 0$ its a maximum.

if $f_{xx} f_{yy} < f_{xy}^2$ it's a saddle point.

Constrained stationary points:

To find stationary values of $f(x,y)$ subject to constraint $g(x,y) = 0$ use the Lagrangian $L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$ subject to $L_x = 0, L_y = 0, L_\lambda = 0$. ~~Here~~ (NB: $L_\lambda = g$.)

Alternatively, if possible, solve the constraint equation, sub into f and differentiate.

The Lagrangian method extends to more independent variables. For additional constraints we will need multiple Lagrangian multipliers.

i.e. $L(x,y,z,\lambda,\mu) = f(x,y,z) - \lambda g(x,y,z) - \mu h(x,y,z)$

- Fields:

A field is a quantity which depends continuously on position.

gradient of a scalar field: $\text{grad}\ \Phi = \underline{i}\dfrac{\partial \Phi}{\partial x} + \underline{j}\dfrac{\partial \Phi}{\partial y} + \underline{k}\dfrac{\partial \Phi}{\partial z}$ $\quad \nabla \Phi$ is common notation.

$d\underline{x} = (dx, dy, dz)$

Directional derivative ~~in dn~~ of field Φ in direction $\underline{t}$ is $\underline{t} \cdot \nabla \Phi = \dfrac{d\Phi}{ds}$

$\Phi = c$ gives a surface, $\nabla \Phi$ is normal vector of that surface.

Line integral of a scalar field:

$$\int_\Gamma \Phi \, ds = \int_{t_1}^{t_2} \Phi(\underline{x}(t)) \left| \frac{d\underline{x}}{dt} \right| dt$$

when the curve Γ is parameterised by $\underline{x}(t)$ & in the interval $[t_1, t_2]$.

Line integral of a vector field:

Given a vector field $\underline{F}(\underline{x})$ and a curve parametrised by variable t in interval $[t_1, t_2]$

$$\int_{\Gamma} \underline{F} \cdot d\underline{x} = \int_{t_1}^{t_2} \underline{F}(\underline{x}(t)) \cdot \frac{d\underline{x}}{dt} \, dt$$

The line integral may depend on the path taken between the two points not just the end points.

Gradient theorem: $\int_{\Gamma} \nabla \Phi \cdot d\underline{x} = \Phi(x_2) - \Phi(x_1)$

i.e. $\nabla \Phi$ is a conservative field.

A conservative field, $\underline{F}$ is a field where:

$\int_{\Gamma} \underline{F} \cdot d\underline{x}$ is independent of path.

$\Leftrightarrow \oint \underline{F} \cdot d\underline{x} = 0 \Leftrightarrow \underline{F} \cdot d\underline{x}$ is exact

$\Leftrightarrow \underline{F} = \text{grad } \Phi$ for some $\Phi \Leftrightarrow \text{curl } \underline{F} = 0$

$\text{div } \underline{F} = \nabla \cdot \underline{F}$ $\quad$ $\text{curl } \underline{F} = \nabla \times \underline{F}$

Vector area: $\underline{S} = \int_S d\underline{S} = \int_S \hat{\underline{n}} \, dS$

Parametrisation of a curved surface:

$\underline{x} = \underline{X}(s,t)$ can parametrize a curve in 3D.

$$d\underline{S} = \left(\frac{\partial \underline{X}}{\partial s} ds\right) \times \left(\frac{\partial \underline{X}}{\partial t} dt\right) = \left(\frac{\partial \underline{X}}{\partial s} \times \frac{\partial \underline{X}}{\partial t}\right) ds\,dt$$

$$\int_S \underline{F} \cdot \underline{n}\, dS = \int_S \underline{F} \cdot d\underline{S} = \int_S F_x\, dy\,dz + \int_S F_y\, dx\,dz + \int_S F_z\, dx\,dy$$ ~~because component~~

$\int_S \underline{F} \cdot d\underline{S}$ is called the flux of $\underline{F}$ through S.

Divergence theorem: $\oint \underline{F} \cdot d\underline{S} = \int_V \operatorname{div} \underline{F}\, dV$ for surface S enclosing volume V.

Stokes' theorem: $\oint_C \underline{F} \cdot d\underline{l} = \int_S \operatorname{curl} \underline{F} \cdot d\underline{S}$ for surface S ending on closed loop C.

Fourier Series:

$f(x)$ periodic with period $2L$ in interval $-L < x < L$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

where $$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} F(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

There is only one Fourier for a given function and periodicity so if there is an easy way to find the Fourier series, do it!

Functions are orthogonal over an interval $[a,b)$ if $\int_a^b f(x)g(x)\,dx = 0$.

Even function: $f(x) = f(-x)$

Odd function: $f(x) = -f(-x)$

~~Gibbs phenomenon:~~
Discontinuity: ~~for~~ the Fourier series of a discontinuous function evaluated at the discontinuity will converge to the average of the limiting values on either side of the discontinuity.